Complex Geometry Exercises

Week 3

Exercise 1. *Let X be a topological space, and A be an abelian group. Consider the presheaves*

- $\mathcal{F}_1(U) = A$ *for all open sets* $U \subseteq X$,
- $\mathcal{F}_2(U) = A$ *for all open sets* $\emptyset \neq U \subseteq X$ *and* $\mathcal{F}_2(\emptyset) = 0$,

with the obvious restriction maps in all cases.

(i) *Show that neither $\mathcal{F}_1$ or $\mathcal{F}_2$ are sheaves.*

(ii) *Show that they have the same étale space, given*

$$\acute{E}t(\mathcal{F}^{const}) = X \times A$$

where A is endowed with the discrete topology, and so $\mathcal{F}_i^+ = \underline{A}$.

(iii) *Compute $\underline{A}(U)$ for a given open set U.*

Exercise 2. *Show that a sequence of sheaves*

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$$

is exact if and only if the corresponding sequence of stalks

$$\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_\S \xrightarrow{\beta_x} \mathcal{H}_x$$

is exact for all $x \in X$.

Exercise 3. *Show that for a topological space $X, U \subset X$, a closed subset with inclusion map $j : U \hookrightarrow X$ and a sheaf $\mathcal{F}$ of abelian groups on U,*

$$H^q(U, \mathcal{F}) \cong H^q(X, j_*\mathcal{F})$$

for all $q > 0$.

(continues on the back)

Exercise 4. *Let X be a complex manifold. We say a sheaf is a* coherent sheaf *if every point $x \in X$ has an open neighbourhood U in X an an exact sequence*

$$\mathcal{O}_X^p|_U \to \mathcal{O}_X^q|_U \to \mathcal{F}|_U \to 0$$

for some natural numbers p and q.

(i) Show that the sheaf of sections of a holomorphic vector bundle is a coherent sheaf.

(ii) Show that a coherent sheaf where we can take $p = 0$ for all points has an associated holomorphic bundle associated to it.

Let $\iota : Z \hookrightarrow X$ be a closed submanifold in X.

(iii) Show that the ideal sheaf of Z, denoted by $\mathcal{I}_Z$, is coherent, where

$$\mathcal{I}_Z(U) = \Big\{ f \in \mathcal{O}_X(U) \Big| \ f|_Z = 0 \Big\},$$

for $U \subseteq X$ open.

(iv) Show that the direct image $\iota^* \mathcal{O}_Z$ is a coherent sheaf.

(v) Prove that there is a short exact sequence of sheaves:

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \iota^* \mathcal{O}_Z \to 0 \ .$$

Exercise 5. *Show that*

$$H^q(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \cong \begin{cases} 0 & \textit{for } q \neq 0 \\ \mathbb{C} & \textit{for } q = 0 \end{cases} .$$